Matrix Description of Waveguide Discontinuities in the Presence of Evanescent Modes

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Summary—The properties of the impedance and scattering matrix describing waveguide discontinuities are examined; both propagating and evanescent modes are considered.

It is shown how different normalization conditions for the normal mode solutions in the guide affect the impedance matrix. A suitable choice of normalization always leads to a symmetric imaginary impedance matrix for a lossless structure.

The scattering matrix is no longer symmetric or unitary. The simple relationship $S=(Z-U)(Z+U)^{-1}$ is shown to hold only under special normalization conditions.

Next the matrices describing a plane of lossless obstacles arranged in a periodic array are examined. A different type of normalization condition must be used here, since the normal modes are orthogonal in the conjugate sense (biorthogonal).

Although the structure is reciprocal, none of the matrices is symmetric. A suitable normalization leads to a skew-hermitian impedance matrix and to a unitary submatrix of the scattering matrix corresponding to propagating modes.

INTRODUCTION

IN TREATING the problem of propagation of electromagnetic waves past waveguide discontinuities it is often convenient to define a set of equivalent voltages and currents, corresponding to linear combinations of incident and reflected wave amplitudes, thereupon reducing it to a circuit problem. The equivalent circuit for the discontinuity lends itself to description in terms of the usual circuit type matrices such as the impedance, admittance, scattering or other matrix.

The properties of the above matrices have been described by Montgomery, Dicke and Purcell,[1] and Kerns[2] when only propagating modes are considered. This is normally the case when the terminal planes on which voltages and currents are defined are chosen far enough from the discontinuity. However, in many cases one cannot neglect the effect of the evanescent modes, as for example when two discontinuities are closely spaced.

The purpose of this paper is to study the properties of the matrices describing waveguide discontinuities, or plane lattices of scatterers, when both propagating and evanescent modes must be considered.

It is important to note that voltages and currents are defined quantities and as such they can be chosen in different ways. Correspondingly the properties of the resulting matrices will be affected by that choice. An alternate way to see this arbitrariness in the definition of the voltages and currents is to note that the normal mode solutions in the waveguide can be normalized in different ways.

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[1] C. G. Montgomery, R. H. Dicke and E. M. Purcell, "Principles of microwave circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Company, Inc., New York, N. Y., vol. 8; 1948.

[2] D. Kerns, "Basis of application of network equations to waveguide problems," *J. Res. NBS*, pp. 515–540; May, 1949.

NORMALIZATION OF MODES

To be specific, consider a rectangular waveguide with a discontinuity extending in the z direction from z_1 to z_2 as in Fig. 1. The transverse fields on the two sides of the discontinuity can be expanded[3] in a set of normal modes

$$\left.\begin{aligned} \boldsymbol{E}_t &= \sum_n a_n \boldsymbol{e}_n e^{-\Gamma_n(z-z_1)} + \sum_n b_n \boldsymbol{e}_n e^{\Gamma_n(z-z_1)} \\ \boldsymbol{H}_t &= \sum_n a_n \boldsymbol{h}_n e^{-\Gamma_n(z-z_1)} - \sum_n b_n \boldsymbol{h}_n e^{\Gamma_n(z-z_1)} \end{aligned}\right\} \text{for } z < z_1$$

$$\left.\begin{aligned} \boldsymbol{E}_t &= \sum c_n \boldsymbol{e}_n e^{-\Gamma_n(z-z_2)} + \sum d_n \boldsymbol{e}_n e^{\Gamma_n(z-z_2)} \\ \boldsymbol{H}_t &= \sum c_n \boldsymbol{h}_n e^{-\Gamma_n(z-z_2)} - \sum d_n \boldsymbol{h}_n e^{\Gamma_n(z-z_2)}. \end{aligned}\right\} \text{for } z > z_2 \quad (1)$$

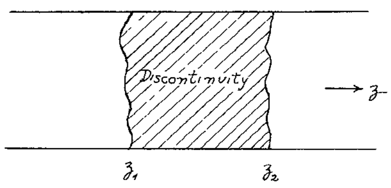

Fig. 1—A general waveguide discontinuity.

$\boldsymbol{e}_n$ and $\boldsymbol{h}_n$ are normal mode functions in the guide and are related by means of a dyadic impedance or admittance

$$\boldsymbol{e}_n = \overline{Z}_n \cdot \boldsymbol{h}_n$$
$$\boldsymbol{h}_n = \overline{Y}_n \cdot \boldsymbol{e}_n \quad (2a)$$

where

$$\overline{Z}_n = Z_n(\boldsymbol{a}_x\boldsymbol{a}_y - \boldsymbol{a}_y\boldsymbol{a}_x)$$
$$\overline{Y}_n = Y_n(\boldsymbol{a}_y\boldsymbol{a}_x - \boldsymbol{a}_x\boldsymbol{a}_y). \quad (2b)$$

Z_n and Y_n are the scalar wave impedance and admittance and are real for propagating modes and imaginary for evanescent modes.

[3] R. E. Collin, "Field Theory of Guided Waves," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 5; 1960

The normal mode functions $\boldsymbol{e}_n$, $\boldsymbol{h}_n$ are orthogonal in a waveguide. Also one of the two functions can always be chosen real. We will assume in the following that $\boldsymbol{e}_n$ is real. These functions can be normalized in a variety of ways; some of the possibilities will now be considered.

Let

$$\iint_S \boldsymbol{e}_n \times \boldsymbol{h}_m \cdot d\boldsymbol{S} = Y_m \iint_S \boldsymbol{e}_n \cdot \boldsymbol{e}_m dS = Y_m N_n \delta_{nm}, \quad (3)$$

where S denotes the waveguide cross section and N_n is a real positive normalization constant, arbitrary as yet.

We choose the voltages to be proportional to the amplitude of the transverse electric field and the currents proportional to the amplitude of the transverse magnetic field, thus

$$\begin{aligned} V_n^+ &= K_{1n}a_n \qquad V_n^- = K_{1n}b_n \\ I_n^+ &= K_{2n}a_n \qquad I_n^- = -K_{2n}b_n. \end{aligned} \quad (4)$$

In order to keep the complex power flow invariant it is necessary that

$$\tfrac{1}{2}V_n^+(I_n^+)^* = \tfrac{1}{2}K_{1n}K_{2n}^* \mid a_n \mid^2 = \tfrac{1}{2}Y_n^*N_n \mid a_n \mid^2 \quad (5a)$$

or

$$K_{1n}K_{2n}^* = Y_n^*N_n. \quad (5b)$$

Also we may choose

$$\frac{V_n^+}{I_n^+} \equiv \frac{K_{1n}}{K_{2n}} = Z_{cn}, \quad (6)$$

where Z_{cn} is any convenient characteristic impedance for the equivalent transmission line.

It can be seen from (5b) that for propagating modes K_{1n} and K_{2n} can both be chosen real since Y_n is real. However, for evanescent modes at least one of the two constants must be chosen imaginary.

One can still choose both K_{1n} and K_{2n} real in all cases provided the definition of complex power flow is modified as follows:

$$\tfrac{1}{2}V_n(I_n^+)^* = \tfrac{1}{2}Y_nN_n \mid a_n \mid^2 = \tfrac{1}{2}K_{1n}K_{2n}^* \mid a_n \mid^2 \quad (7a)$$

for propagating modes,

$$\pm\tfrac{1}{2}jV_n(I_n^+)^* = \tfrac{1}{2}Y_n^*N_n \mid a_n \mid^2 = \pm\tfrac{1}{2}jK_{1n}K_{2n}^* \mid a_n \mid^2 \quad (7b)$$

for evanescent modes.

In (7b) the upper sign holds for H modes and the lower sign for E modes.

Before actually choosing a specific normalization we will first derive some general properties of the impedance matrix.

Impedance Matrix

We will number the modes on the two sides of the discontinuity in consecutive order, that is, we can define a voltage vector and current vector (column matrix)

$$\begin{aligned} V &= V^+ + V^- \\ I &= I^+ - I^- \end{aligned} \quad (8)$$

where

$$V^+ = \begin{bmatrix} V_1^+ \\ V_2^+ \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad V^- = \begin{bmatrix} V_1^- \\ V_2^- \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

and similarly for the currents. In view of the linearity of Maxwell's equations we have

$$V = ZI \quad (9)$$

where Z is the impedance matrix.

Consider first two independent solutions to Maxwell's equations satisfying boundary conditions in the guide. The following relation then holds in a reciprocal medium:

$$\nabla\cdot(\boldsymbol{E}^1 \times \boldsymbol{H}^2 - \boldsymbol{E}^2 \times \boldsymbol{H}^1) = 0. \quad (10)$$

The superscripts refer to the two independent solutions. When this relation is integrated over a region enclosing the discontinuity we obtain

$$\begin{aligned} &\iiint_V \nabla\cdot(\boldsymbol{E}^1 \times \boldsymbol{H}^2 - \boldsymbol{E}^2 \times \boldsymbol{H}^1)dv \\ &= \iint_S (\boldsymbol{E}^1 \times \boldsymbol{H}^2 - \boldsymbol{E}^2 \times \boldsymbol{H}^1)\cdot d\boldsymbol{S} \\ &= \sum_n (V_n^1 I_n^2 - V_n^2 I_n^1)\frac{Y_nN_n}{K_{1n}K_{2n}} = 0 \end{aligned} \quad (11)$$

where it was assumed that the transverse fields have been expanded in normal modes with the normalization as given by (3).

Consider now two conditions for the terminal planes:

Condition 1)

$$V_n^1 = 0 \qquad n \neq i \qquad I_i^2 = Y_{ij}V_j^2$$

Condition 2)

$$V_n^2 = 0 \qquad n \neq j \qquad I_j^1 = Y_{ji}V_i^1.$$

Then it follows from (11) that

$$Y_{ij}\frac{Y_iN_i}{K_{1i}K_{2i}} = Y_{ji}\frac{Y_jN_j}{K_{1j}K_{2j}} \quad (12)$$

where Y_{ij} and Y_{ji} are elements of the admittance matrix. The admittance matrix, and thus the impedance matrix, will be symmetric if

$$\frac{Y_iN_i}{K_{1i}K_{2i}} = \frac{Y_jN_j}{K_{1j}K_{2j}}. \quad (13)$$

At this point we will make the choice of normalization. Consider the following cases:

Case I)

$$\iint_S \boldsymbol{e}_n \cdot \boldsymbol{e}_n dS = N_n = 1. \quad (14)$$

Using (5b) we must have

$$K_{1n}K_{2n}^* = Y_n^*.$$

We can then choose $K_{1n}=1$ and $K_{2n}=Y_n$. With this choice (13) is satisfied and thus

$$Y_{ij} = Y_{ji} \quad \text{and also} \quad Z_{ij} = Z_{ji}.$$

Case II)

$$\iint_S e_n \cdot e_n dS = N_n = \begin{cases} \dfrac{1}{Y_n} & \text{for propagating modes} \\ \dfrac{\pm j}{Y_n} & \text{for evanescent modes} \end{cases} \quad (15)$$

The (+) and (−) signs correspond to E and H modes. Again using (5b) we must have

$$K_{1n}K_{2n}^* = 1 \quad \text{for propagating modes,}$$

$$K_{1n}K_{2n}^* = \mp j \quad \text{for evanescent modes.}$$

This then allows us to choose

$$K_{1n} = K_{2n} = 1 \quad \text{for propagating modes,}$$

$$K_{1n} = 1, \qquad K_{2n} = \pm j \quad \text{for evanescent modes.}$$

With that choice (13) is satisfied and both the admittance and impedance matrix are again symmetric.

Further properties of the impedance matrix can be derived from the energy condition. Integrating the complex Poynting vector over a region containing the discontinuity we obtain

$$\frac{1}{2}\iint_S \boldsymbol{E} \times \boldsymbol{H}^* \cdot d\boldsymbol{S} = P + 2j\omega(W_m - W_e) = \frac{1}{2}\tilde{V}I^* \quad (16)$$

where the tilde stands for the transposed matrix. If the structure is lossless

$$P = 0 = \tfrac{1}{2}\,\mathrm{Re}\,(\tilde{V}I^*) = \tfrac{1}{4}[\tilde{I}^*(Z + \tilde{Z}^*)I].$$

Then

$$Z + Z^* = 0 \quad (17)$$

since $Z = \tilde{Z}$.

We conclude that under normalization condition (14) or (15) with the definition of complex power in the sense of (5b) the impedance matrix of a lossless discontinuity is symmetric and imaginary. The same holds for the admittance matrix.

It is sometimes convenient to choose the characteristic impedances of the equivalent transmission lines as unity for all modes. As we have seen before (5b) does not allow us that choice unless we resort to the modified definition of complex power flow as given by (7a) and (7b). Using (15) we obtain

$$K_{1n}K_{2n}^* = 1$$

for both propagating and evanescent modes. It is now possible to choose

$$K_{1n} = K_{2n} = 1.$$

The properties of the impedance matrix are now somewhat different. It is convenient to partition the voltage, current and impedance matrices as follows:

$$\begin{bmatrix} V_p \\ V_e \end{bmatrix} = \begin{bmatrix} [Z_{pp}] & [Z_{pe}] \\ [Z_{ep}] & [Z_{ee}] \end{bmatrix} \begin{bmatrix} I_p \\ I_e \end{bmatrix} \quad (18)$$

where the subscripts p and e refer to propagating and evanescent modes.

The reciprocity relation (11) now becomes

$$\tilde{V}_p{}^1 I_p{}^2 - \tilde{V}_p{}^2 I_p{}^1 \pm j[\tilde{V}_e{}^1 I_e{}^2 - \tilde{V}_e{}^2 I_e{}^1] = 0. \quad (19)$$

The upper sign holds for E modes and the lower sign for H modes. Consider now the following conditions for the terminal planes:

Condition 1)

$$[I_p{}^1] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_i{}^1 \\ \vdots \\ 0 \end{bmatrix}$$

$$[I_p{}^2] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_j{}^2 \\ 0 \end{bmatrix}$$

With

$$[V_p{}^1] = Z_{ji}[I_p{}^1] \qquad [V_p{}^2] = Z_{ij}[I_p{}^2]$$

we obtain

$$Z_{ij} = Z_{ji}.$$

The submatrix $[Z_{pp}]$ is symmetric. We now assume that only E or H modes are present, otherwise, because of the sign ambiguity in (19) further partitioning of the impedance matrix is necessary to obtain its properties.

Condition 2)

$$[I_e{}^1] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_i{}^1 \\ \vdots \\ 0 \end{bmatrix} \qquad [I_e{}^2] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_j{}^2 \\ \vdots \\ 0 \end{bmatrix}$$

We then obtain

$$Z_{ij} = Z_{ji} \quad \text{for the } [Z_{ee}] \text{ submatrix.}$$

Condition 3)

$$[I_p^1] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_{pi}^1 \\ \vdots \\ 0 \end{bmatrix} \qquad [I_e^2] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_{ej}^2 \\ \vdots \\ 0 \end{bmatrix}$$

With

$$[V_p^1] = Z_{pe}^{ij}[I_e^2] \qquad [V_e^2] = Z_{ep}^{ji}[I_p^1]$$

we obtain

$$Z_{pe}^{ij} \pm jZ_{ep}^{ji} = 0 \tag{20}$$

or more generally

$$[Z_{ep}] = \pm j[\tilde{Z}_{pe}].$$

If we now impose the energy condition for a lossless structure on the impedance matrix we have

$$P = 0 = \text{Re}\,[\tilde{I}_p^* V_p \pm j\tilde{I}_e^* V_e] = \text{Re}\,[\tilde{I}_p^* Z_{pp} I_p] + \text{Re}\,[\pm \tilde{I}_e^* Z_{ee} I_e] + \text{Re}\,[\tilde{I}_p^* Z_{pe} I_e] + \text{Re}\,[\pm j\tilde{I}_e^* Z_{ep} I_p].$$

Therefore

$$[Z_{pp}] + [Z_{pp}^*] = 0 \qquad [Z_{pp}] \text{ is imaginary}$$
$$[Z_{ee}] - [Z_{ee}^*] = 0 \qquad [Z_{ee}] \text{ is real.}$$

Also

$$\tilde{I}_p^*[Z_{pe} \mp j\tilde{Z}_{ep}^*]I_e = 0.$$

Therefore

$$[Z_{pe} \mp j\tilde{Z}_{ep}^*] = 0 \tag{21}$$

but

$$[Z_{pe} \pm j\tilde{Z}_{ep}] = 0.$$

Then

$$[Z_{pe}] = [\pm j\tilde{Z}_{ep}^*] = [\mp j\tilde{Z}_{ep}]$$
$$[Z_{ep}^* + Z_{ep}] = 0.$$

Therefore

$$[Z_{ep}] \text{ is imaginary}$$
$$[Z_{pe}] \text{ is real.}$$

Scattering Matrix

An alternate description of waveguide discontinuities is by means of the scattering matrix.

We define the matrix by

$$V^- = SV^+. \tag{22}$$

If we choose $K_{1n} = K_{2n} = 1$ for all modes then

$$V = V^+ + V^- = Z(I^+ + I^-) = Z(V^+ - V^-)$$
$$(Z - U)V^+ = (Z + U)V^-$$

and

$$S = (Z + U)^{-1}(Z - U). \tag{23}$$

The matrix S given by (23) is not symmetric since Z is not symmetric for this case. Also S is not a unitary matrix any more.

Conditions (14) and (15) do not lead to a simple relationship between the impedance matrix and the scattering matrix, because the characteristic impedances of the equivalent transmission lines are real for propagating modes and imaginary for evanescent modes. The properties of the scattering matrix as given by (23) will be studied in a later part of this paper.

Matrices of Periodic Arrays of Scatterers

We now consider a plane of lossless obstacles arranged in a periodic array; no longer is it possible here to use normalization conditions (14) or (15) because the normal mode solutions in the structure are always complex such as to satisfy periodic boundary conditions in the transverse plane.

The following normalization procedure, in the conjugate sense, can be used for both a lossless discontinuity in a guide and an array of lossless scatterers.

Let

$$\iint_S \boldsymbol{e}_n \times \boldsymbol{h}_m^* \cdot \boldsymbol{dS} = Y_m^* \iint_S \boldsymbol{e}_n \cdot \boldsymbol{e}_m^* dS = Y_m^* N_n \delta_{nm} \tag{24}$$

where S denotes the cross section of the unit cell in the structure and N_n is a positive real normalization constant.

It should be noted that it is still possible to normalize the modes in a lossless periodic structure as in (14) or (15) provided one restricts propagation to incidence normal to the plane. That is indeed the case since the E and H modes are orthogonal in a periodic structure of this sort just like in a waveguide. Again consider two cases:

Case I)

$$\iint_S \boldsymbol{e}_n \cdot \boldsymbol{e}_n^* dS = N_n = 1 \tag{25}$$

Case II)

$$\iint_S \boldsymbol{e}_n \cdot \boldsymbol{e}_n^* dS = \begin{cases} \dfrac{1}{Y_n} & \text{for propagating modes} \\[2ex] \mp \dfrac{j}{Y_n} & \text{for evanescent modes} \end{cases} \cdot \tag{26}$$

The upper sign holds for E modes and the lower for H modes. A reciprocity relation can be written for the field in the structure as follows:

$$\nabla \cdot (\boldsymbol{E}^1 \times \boldsymbol{H}^{*2} + \boldsymbol{E}^{*2} \times \boldsymbol{H}^1) = 0. \tag{27}$$

However, this relation assumes a lossless structure. Therefore, it is easier to use directly the energy condition.

The complex Poynting vector integrated over the terminal planes of a unit cell yields

$$\frac{1}{2}\iint \boldsymbol{E} \times \boldsymbol{H}^* \cdot d\boldsymbol{S} = 2j\omega(W_m - W_e) = \frac{1}{2}\tilde{V}I^* \quad (28)$$

if one defines complex power in the sense of (5b) with normalization given by (25) or (26).

The structure is lossless so

$$P = 0 = \tfrac{1}{2}\,\mathrm{Re}\,(\tilde{V}I^*) = \tfrac{1}{4}(\tilde{I}^*ZI + \tilde{I}^*\tilde{Z}^*I)$$

or $Z+\tilde{Z}^*=0$. The impedance matrix is thus skew-hermitian.

However, if normalization condition (26) is used together with the power definition (7a) and (7b) the complex integration yields

$$\mathrm{Re}\,(\tilde{I}_p^*Z_{pp}I_p) + \mathrm{Re}\,(\pm j\tilde{I}_e^*Z_{ee}I_e) + \mathrm{Re}\,(\tilde{I}_p^*Z_{pe}I_e) + \mathrm{Re}\,(\pm j\tilde{I}_e^*Z_{ep}I_p) = 0$$

where $[Z]$ is partitioned as in (18). We must then have

$$[Z_{pp}] + [\tilde{Z}_{pp}^*] = 0 \quad \text{or} \quad [Z_{pp}] \text{ is skew-hermitian}$$

$$[Z_{ee}] - [\tilde{Z}_{ee}^*] = 0 \quad \text{or} \quad [Z_{ee}] \text{ is hermitian}$$

$$[Z_{pe}] \mp [j\tilde{Z}_{ep}^*] = 0.$$

This impedance matrix is related to the scattering matrix by means of (23). No simple relationship exists between the S and Z matrices obtained from any other normalization.

To obtain the properties of the scattering matrix we partition it as follows:

$$\begin{bmatrix} V_p^- \\ V_e^- \end{bmatrix} = \begin{bmatrix} [S_{pp}] & [S_{pe}] \\ [S_{ep}] & [S_{ee}] \end{bmatrix} \begin{bmatrix} V_p^+ \\ V_e^+ \end{bmatrix}. \quad (29)$$

The real power is given by

$$P = 0 = (\tilde{V}_p^+)^*V_p^+ - (\tilde{V}_p^-)\ V_p^- \pm j[(V_e^+)^*V_e^- - (\tilde{V}_e^-)^*V_e^+] = 0 \quad (30)$$

for a lossless structure.

Using (29) this becomes

$$(\tilde{V}_p^+)^*[U - \tilde{S}_{pp}^*S_{pp}]V_p^+ - (\tilde{V}_e^+)^*[\tilde{S}_{pe}^*S_{pe} \mp j(S_{ee} - \tilde{S}_{ee}^*)]V_e^+ - (\tilde{V}_e^+)^*[\tilde{S}_{pe}^*S_{pp} \mp jS_{ep}]V_p^+ - (\tilde{V}_p^+)^*[\tilde{S}_{pp}^*S_{pe} \pm j\tilde{S}_{ep}^*]V_e^+ = 0.$$

This then yields

$$[\tilde{S}_{pp}^*][S_{pp}] = [U] \quad \text{or} \quad [S_{pp}] \text{ is unitary}$$

$$[S_{ee} - \tilde{S}_{ee}^*] = \pm j[\tilde{S}_{pe}^*][S_{pe}]$$

$$[\tilde{S}_{pp}^*][S_{pe}] \pm [jS_{ep}] = 0.$$

The above scattering matrix is defined with normalization given by (26) and power definition given by (7a) and (7b).

Conclusion

In conclusion we have shown that for a lossless discontinuity in a waveguide it is always possible to choose the equivalent voltages and currents such as to obtain an impedance matrix which is symmetric and imaginary just like in the case when only propagating modes are present. For this case however the characteristic impedance of the equivalent transmission lines are real for propagating modes and imaginary for the evanescent modes. This normalization is preferable when both E and H modes are present because it avoids the sign ambiguity of (19) in the derivation of the properties of the impedance matrix.

If one tries to make all characteristic impedances of the transmission lines real for all modes the corresponding impedance matrix is no longer symmetric or imaginary.

For a periodic array of scatterers it is possible to choose voltages and currents such that the impedance matrix is skew-hermitian. Although the structure is reciprocal the impedance matrix is not symmetric.

The scattering matrix is no longer symmetric or unitary in either case; it is only possible to obtain a unitary submatrix corresponding to propagating modes.

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